# Math 259A Lecture 15 Notes

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November 1, 2019

## 1 Geometry of Projections and Classification of von Neumann Algebras

### 1.1 Closed graph operators

**Definition 1.1.** A **closed graph operator** is a linear operator $T : D(T) \to H$, where $D(T) \subseteq H$ is a dense subspace, such that the graph of $T$, $G_T = \{(\xi, T\xi) : \xi \in D(T)\} \subseteq H \times H$, is closed (i.e. whenever $\xi_n \to 0$ and $T\xi_n \to \eta$, then $\eta = 0$).

**Example 1.1.** Let $\ell^2(\mathbb{N})$ have its usual orthonormal basis $\xi_n$. Now define $T_0(\sum c_n\xi_n) = \sum nc_n\xi_n$, which is defined on $D(T_0) = H_0$, the space of finite sums. Now consider $\overline{G_{T_0}}$; there exists some $T$ such that $G_T = \overline{G_{T_0}}$. The space of sequences $\sum_{n=1}^\infty c_n\xi_n$ with $\sum_{n=1}^\infty |nc_n|^2 < \infty$ is $D(T)$.

### 1.2 More geometry of projections

Recall some definitions from last time:

**Definition 1.2.** $e \in P(M)$ is **abelian** if $eMe$ is abelian.

**Definition 1.3.** $e \in P(M)$ is a **finite projection** if whenever $f \leq e$ and $f \sim e$, $f = e$.

**Definition 1.4.** A von Neumann algebra $M$ is **finite** if 1 is a finite projection (i.e. any isometry is necessarily a unitary).

**Definition 1.5.** $e \in P(M)$ is **properly infinite** if $e$ has no direct summands in $M$ that are finite, i.e. if $p \in P(M) \cap Z(M)$ with $pe$ finite, then $pe = 0$.

**Lemma 1.1.** *Let $e \leq f \in P(M)$ be abelian. Then*

1. *$e = z(e)f$.*
2. *If $z(e) \leq z(f)$, then $e \prec f$.*

**Remark 1.1.** We always have that if $e \prec f$, then $z(e) \leq z(f)$.

**Lemma 1.2.** *If $e \in P(M)$ contains no abelian projection (i.e. if $f \leq e$ is abelian, $f = 0$), then there exist $e_1, e_2 \in P(M)$ such that $e_1 \sim e_2$, and $e_1 + e_2 = e$.*

*Proof.* Take maximal (with respect to inclusion) mutually orthonormal sets $\{e_i\}_I, \{f_i\}_I$ under $e$. We claim that $\sum_{i\in I} e_i + \sum_{i \in I} f_i = e$; if we call this $p$ and $e - p \neq 0$, then $(e-p)M(e-p)$ is not abelian. Then there exists an $e_0' \sim f_0' \neq 0$ that we can add to the orthonormal sets, contradicting maximality. □

**Lemma 1.3.** *A projection $e \in P(M)$ is properly infinite if and only if $e = \sum_{n=1}^{\infty} e_n$ with $e_n \sim e$.*

*Proof.* Use Vold's decomposition. Start by building a family $\{f_n\}$ of mutually orthogonal, mutually equivalent operators. If, say, $e = \sum e_n^0$ with $e_n^0 \sim e_m^0$, then split $\mathbb{N} = \bigcup_{m=1}^{\infty} N_m$ with $|N_m| = \infty$. Then define $e_m = \sum_{k \in N_k} e_k^0$. This is equivalent to $\sum_{m\in\mathbb{N}} e_n^0 = e$. □

**Definition 1.6.** A projection $e \in P(M)$. is of **countable type** if when $\{e_i\}_{i\in I}$ are mutually orthogonal and $\leq e$, then $|I|$ is countable.

**Example 1.2.** $\mathcal{B}(\ell^2(\mathbb{N}))$ only has projections of countable type, but $\mathcal{B}(\ell^2(\mathbb{R}))$ has projections not of countable type.

**Lemma 1.4.** *Let $e, f \in P(M)$, let $e$ be of countable type, and let $f$ be properly infinite. If $z(e) \leq z(f)$, then $e \prec f$.*

*Proof.* Take $\{e_i\}_{i\in I}$ mutually orthogonal, $\leq e$, and such that $e_i \prec f$ for all $i$; take a maximal such family with respect to inclusion. We claim that $\sum_i e_i = e$. Indeed, if $p := e - \sum_i e_i \neq 0$, then if $pMf \neq 0$, we contradict maximality: taking $x$ such that $pxf \neq 0$, we get that $\ell(pxf) \sim r(pxf)$. If $pMf = 0$, then $z(p) \leq z(f) = 0$. So $I$ is countable; that is, $e = \sum_n e_n$ with $e_n \prec f$ for all $n$. But then by induction on $n$, one builds projections $f_n \leq f$ such that $f_n$ are mutually orthogonal and $e_n \sim f_n$. Now use the previous lemma. □

## 1.3 Classification of von Neumann algebras

**Definition 1.7.** A von Neumann algebra $M$ is **semifinite** if $1_M = \bigvee_i e_i$ with $e_i$ finite.

**Example 1.3.** $\mathcal{B}(\ell^2(\mathbb{N}))$ is semifinite.

**Definition 1.8.** A von Neumann algebra $M$ is **type I** if $1_M = \bigvee_i e_i$ with $e_i$ abelian.

**Example 1.4.** $\mathcal{B}(\ell^2(I))$ is of type I for any $I$.

**Definition 1.9.** A von Neumann algebra $M$ is **type II** if it is semifinite and has no abelian projections.

So far in this course, we have no examples of type II von Neumann algebras.

**Definition 1.10.** A von Neumann algebra $M$ is **type II** has no finite projections.

We have no examples yet of this, either.

**Definition 1.11.** A von Neumann algebra $M$ is **type I finite** if it is of type I and finite. $M$ is of **type I infinite** if it is of type I but has no central finite projection.

**Example 1.5.** $\mathcal{B}(\ell^2(\mathbb{N}))$ is of type 1 finite. Type 1 infinite algebras looks like $\bigoplus_i \mathcal{B}(\ell^2(J_i)) \otimes L^\infty(X_i)$, where $|J_i| = \infty$.

We can state similar definitions for type II algebras. Here is the key lemma:

**Lemma 1.5.** *Let $\{e_i\} \subseteq P(M)$ be mutually orhogonal with mutually orthogonal central supports $z(e_i)$.*

1. *If all $e_i$ are abelian, then $\sum_i e_i$ is abelian.*
2. *If all $e_i$ are finite, then $\sum_i e_i$ is finite.*

**Theorem 1.1.** *Let $M$ be a von Neumann algebras. There exist $p_1, p_2, p_3, p_4, p_5 \in P(M) \cap Z(M)$ with $\sum_{i=1}^5 p_i = 1$ such that $Mp_1$ is of type I finite, $Mp_2$ is type I infinite, $Mp_3$ is type $II_1$, $Mp_4$ is type II infinite, and $Mp_5$ is type III. So if $M$ is a factor then it is either isomorphic to $M_n(\mathbb{C})$ for some $n$, $\mathcal{B}(\ell^2(I))$, a type I, and type II, or a type III.*